Improving Smallholder Welfare While Preserving Natural Forest: Intensification vs. Deforestation

Xavier Warnes, Dan Iancu, Erica Plambeck
Stanford University

Increasing the welfare of smallholder farmers in developing countries plays a crucial role in the global effort to reduce worldwide poverty and hunger. On the one hand, smallholders represent a large proportion of the world's poor and, on the other, they produce the majority of the food consumed in developing countries. This realization has led governments and organizations around the world to implement policies aimed at increasing farmers' yields. Although most of these policies have resulted in welfare increases, the environmental effects have been varied. While in many settings intensification policies have been linked to a decrease in deforestation, in many other settings the reverse is true. In this chapter we propose a novel explanation of these seemingly contradictory results. We achieve this through studying a detailed operational model of a farmer's dynamic decisions of land-clearing and production. We show the importance of considering the interaction between random production costs and liquidity constraints faced by smallholder farmers. These two elements are key to our main result: a reduction in the cost of intensification can lead to *lower* deforestation rates when the variation in production costs is high enough compared to the cost of intensification. Alternatively, the same reduction in the cost of intensification may lead to *higher* deforestation rates if the variation in production costs is low enough compared to the cost of intensification. This result helps explain the discrepancies seen in practices and may allow policy makers to better target interventions in order to achieve win-win situations: improvement of smallholder welfare and protection of the natural forest.

Key words:

History:

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1. Introduction

Out of the 1.4 billion people living on less than US$1.25 per day, one billion are smallholder farmers working on land plots smaller than 2 hectares (Rapsomanikis 2015). Yet, at the same time, over 80 per cent of the food consumed in a large part of the developing world is produced by smallholders (IFAD 2013). This puts smallholders at the center of the global efforts to both reduce poverty and increase agricultural production. The latter being ever more important given the rising food demand that is roughly expected to double by 2050 (Tilman et al. 2011). Motivated by these goals and inspired by the Asian Green Revolution of the 60's and 70's (Hazell 2009), governments and

NGOs have been actively implementing programs to increase agricultural productivity (see e.g., Djurfeldt et al. 2005, Rashid et al. 2013).

Many of these programs have had some success in increasing the total agricultural productivity. However, in many cases, they have also contributed to the increasing rates of tropical deforestation (see § 1.2). Indeed, agricultural expansion has been widely recognized as one of the main causes of tropical deforestation (Geist and Lambin 2002, Kissinger et al. 2012), which in turn is one of the leading causes of anthropogenic Green House Gas emissions (Houghton 2012).

At the same time, the rising temperatures and increased variability of weather events caused by climate change directly affects smallholders (Nelson et al. 2014). Faced with little capital and higher variability in their costs, smallholders turn to land expansion. This makes it even more important to understand the following question: can agricultural *intensification* be achieved while avoiding *deforestation*?

In order to better understand the answer to this question, we present a dynamic model of farmer operations under liquidity constraints and random production costs. We show that the combination of these two factors plays a major role in determining when intensification—defined as any increase in agricultural productivity—will exacerbate or mitigate deforestation. In particular, we show that reducing the cost of intensification can either increase or decrease the deforestation pressure, depending on how large the marginal intensification costs are compared to the variation in production costs.

1.1. Main Contribution

We develop a general dynamic model of a farmer's operations, allowing for both productive and land-clearing decisions under liquidity constraints. This allows us to study the effects of changing the farmer's cost structure on their optimal decisions. Our model confirms many previously found results on how farmer welfare increases when reducing production costs (both the average and variability of costs), and when improving access to loans (e.g., Asfaw et al. 2012). Furthermore, we show that although all of these changes would increase the total intensification effort of farmers, they would also lead to a higher rate of deforestation.

We show that, surprisingly, directly reducing the cost of intensification may have different effects on the rate of deforestation: if the intensification cost is low compared to the production cost variability, reducing the intensification cost *reduces* the deforestation pressure. On the other hand, if the intensification cost is high, a reduction would *increase* the pressure. This result helps to shed light on the contradicting empirical evidence linking intensification promotion and deforestation (see §1.3).

Central to our results is our consideration of random production costs that are directly proportional to the total productive land. These random shocks occur frequently in practice due to uncertainty when bringing products to market. For instance, transportation costs may be greatly affected by weather conditions when roads are not paved, or labor costs may be higher than expected during harvest season. These risks in the total production costs combine with liquidity constraints to generate a downward pressure on the deforestation rate: faced with a high risk of having to borrow at high interest rates, farmers react to a reduction in the cost of intensification by increasing their rate of production in a smaller productive area.

Our results highlight the importance of considering the specific operational context when designing policy interventions. As has been shown many times in practice, the indiscriminate application of policies can have significant negative consequences. Our model helps policy makers understand the relationship between intensification and deforestation by categorizing farmers and communities of farmers based on their intrinsic characteristics.

1.2. Related Literature

The Environmental Science literature has widely documented that agricultural expansion is the dominant driver of illegal deforestation in developing countries (e.g., Carlson et al. 2018, Geist and Lambin 2002). Although there are many proximate causes of deforestation processes, such as economic and institutional factors, the question of whether yield increases is one of these causes is still very much under debate. There is a significant amount of empirical evidence showing how yield increases may lead to both lower rates of deforestation and, on the contrary, higher deforestation rates (see §1.3 for a summary of some of these settings). Stemming from these empirical observations, there has been extensive work in the Agricultural Economics literature to shed light on these seemingly contradictory effects (see Angelsen and Kaimowitz 2001 and Angelsen and Kaimowitz 1999 for excellent reviews of these models). Most of the explanations put forth in this body of work can be broadly categorized into three types: labor supply driven (e.g., Maertens et al. 2006, Shively and Pagiola 2004), driven by the elasticity of demand for the agricultural products (e.g., Jayasuriya et al. 2001), or driven by the different types of utility functions of the farmers (e.g., Angelsen et al. 2001). Our work adds to this discussion by considering the role of random production costs paired with liquidity constraints and showing how these two factors play a major role in determining how intensification will affect deforestation.

To develop our model of farmer operations, we use insights from development economics, operations-finance, and sustainable-operations. From the latter, our model generalizes the farmer dynamic model presented in de Zegher et al. (2018), by allowing for dynamic deforestation decisions and generalizing the concave production functions used. From the operations-finance literature, our formulation resembles the models of dynamic production decisions under limited cash inventory, such as Li et al. (2013) and Ning and Sobel (2017). In our model, farmers experience Guassian production cost shocks, that drive them to informally borrow at high rates from specific agents within their community, this is in line with findings from the Development Economics literature (e.g., Collins et al. 2009). Economists have documented that smallholder farmers have limited access to the formal financial system, and rely on informal loans within their communities (Duflo and Banerjee 2011), paying interest rates that increase in the size of the loans. To model these increasing interest payments we adopt an exponential function (see e.g., Ghosh et al. 2000).

In the Environmental Economics and Mechanism Design literature there has been recent interest in designing optimal mechanisms to halt deforestation and preserve natural ecosystems (see, e.g., Mason and Plantinga 2013, Li et al. 2020). Most of these works have been focused on the design of Payments for Ecosystem Services, and not on the detailed operations of farmers. Although in our work we aim at establishing mechanisms for forest protection, we concentrate on the farmers' production operations, and not on the principal-agent problems that arise from the possible conservation mechanisms.

Our work connects as well with the growing body of work in the operations management literature aimed at improving farmers' welfare and social welfare in agricultural supply chains (see Bouchery et al. 2016 and Kalkanci et al. 2019 for reviews). Several recent studies have focused on the production operations of farmers (e.g., Dawande et al. 2013, Boyabatlı et al. 2019, Federgruen et al. 2019, Hu et al. 2019, Levi et al. 2020), as well as the effects of different subsidy schemes on farmer's decisions (e.g., Chintapalli and Tang 2018, Alizamir et al. 2019, Akkaya et al. 2021). In our work we not only examine mechanisms that lead to higher farmer welfare, but study dynamic deforestation decisions.

1.3. Examples

In this section we present several empirical examples of how promoting intensification has lead to both increase and decrease of deforestation pressures. The purpose of this section is to illustrate how our result connects to the empirical literature, but it is not an exhaustive review of all such works (see Angelsen and Kaimowitz (2001) and references therein for a broader review).

Governments and NGOs routinely implement two types of programs to incentivize intensification; the first aimed at reducing the cost of inputs, such as fertilizers or pesticides (see, e.g., Pelletier et al. 2020), the second aimed at decreasing the cost of new technology adoptions, such as higher yield seeds or better agricultural practices (see, e.g., Garrett et al. 2013).

In the case of reducing the cost of inputs, this is usually done through either subsidies or low interest rate credits. This is the case for the fertilizer subsidy programs implemented in Zambia and Malawi (see Pelletier et al. 2020, Abman and Carney 2020). Interestingly, although both programs were implemented with similar goals and were indeed successful in increasing yields and farmer welfare, the evidence suggests that in Zambia increased fertilizer use was weakly linked to increased deforestation, while in Malawi, the reverse was found.

In order to reduce the cost of new technology adoption, interventions usually include a combination of training and subsidies for the purchase of new improved inputs (e.g., better seeds). In the case of the Brazilian policy of providing low interest rate credits for the purchase of higher producing soybeans (that was put into place at the end of the 20th century) the results suggest that the improvement in yields led to a higher deforestation rate of the Amazonian forest (Garrett et al. 2013). In Malawi and Zambia, together with the fertilizer programs, the governments implemented high subsidies for the purchase of high-yield maize seeds (Pelletier et al. 2020, Abman and Carney 2020). In contrast to what happened when subsidizing fertilizer, the yield increase caused by the new maize-seeds reduced deforestation in both countries. This same effect was observed in Bangladesh, after government programs subsidized higher yielding crops (Aravindakshan et al. 2021).

Finally, examples of training in better agricultural practices and technologies can be found Indonesia and Malaysia (e.g., Maertens et al. 2006, Villoria et al. 2013). In the case of Indonesia, Maertens et al. (2006) differentiate between the effects observed by yield-saving technologies (that reduced deforestation) and labor-saving technologies (that increased deforestation). While Villoria et al. (2013) observed an increase in deforestation related to higher yields in the oil-palm value chain. Table 1 provides a summary of these varied empirical findings.

Country	Type of Intervention	Does Intensification Lead to Deforestation?	Reference
Brazil	Credits for purchasing better yielding soybeans.	Yes,higher yields led to increased deforestation in the absence of strong regulations.	Garrett et al. (2013)
Indonesia (Lore Lindu)	Introduction of labor saving and yield increasing technologies.	Labor saving technologies increased deforestation, yield increasing technologies reduced deforestation.	Maertens et al. (2006)
Indonesia and Malaysia	Training in better agricultural practices and technologies.	Yes, higher yields were associated with higher deforestation.	Villoria et al. (2013)
Zambia	Subsidy of fertilizer and improved maize seeds.	Fertilizer subsidy was weakly linked to increased deforestation; Improved seeds use was linked to a decrease in deforestation.	Pelletier et al. (2020)
Malawi	Subsidies for fertilizer and higher yield seeds.	No, lower rates of deforestation were observed.	Abman and Carney (2020)
Bangladesh	Introduction of higher yielding crops.	No, lower rates of deforestation were observed.	Aravindakshan et al. (2021)

Table 1 Empirical evidence on the Intensification-Deforestation connection. Summary of some of the many works showing how intensification can either cause or prevent deforestation.

2. Model Formulation

We model the operations of a liquidity constrained smallholder-farmer that at every period faces a random production cost shock and must decide on both consumption and production decisions. At the start of each discrete production period n, the farmer observes the current market price p_n, exogenously fixed, and makes three decisions: the rate of consumption c_n, the total amount of land to expand l_n^d, and the total amount of productive inputs or technologies used per unit of time and per unit of land, y_n (henceforth we shall refer to this term as the *productive expenditure rate*). The productive-expenditure may represent the total amount of certain inputs used (e.g, fertilizer, pesticides, insecticides, or labor for preparing the field, planting, and weeding) or the level of adoption of productive technologies (e.g., higher-yield seeds, increased water use), and is characterized by having a concave increasing effect on the total yield.

Formally, consider the time interval $[0, D]$ divided into production periods of length τ. Let the n-th time period be the interval $[(n-1)\tau, n\tau]$ (we assume for simplicity that D is a multiple of τ), and let $N := \frac{D}{\tau}$ be the total number of periods. The timing of decisions is then as follows: at the beginning of the n-th period (i.e., time $(n-1)\tau$), the farmer observes the market price p_n, which we assume comes from an exogenous random process, and that will be paid at the end of the n-th production cycle (i.e., time $n\tau$). Additionally, at the start of the n-th period, the farmer has a cash position x_n, and total productive land ℓ_n. At this time the farmer decides the consumption rate per unit of time c_n, the productive-expenditure rate $y_n \geq 0$, and the total land to expand during period n, ℓ_n^d. The productive-expenditure rate will generate a rate of production given by $(y_n)^\lambda \ell_n$ during period n, for a fixed $0 \leq \lambda \leq 1$. This leads to a total production of $(y_n)^\lambda \ell_n \tau$, during the nth period. Although the land expansion occurs at the start of the period, the land expanded will not become productive until the next period (i.e., time $n\tau$). The total consumption during period n will be $c_n \tau$. At the end of each period, the farmer receives a payment of $(y_n)^\lambda \ell_n \tau p_n$. The land expansion at the start of the period will have a total cost of $(\ell_n^d)^+ d$, where d is the combination of the cost of clearing and making the new land productive. During period n, the farmer will face a total production cost of

$$((y_n)^\lambda \mathcal{W}_n + y_n q)\ell_n \tau. \tag{1}$$

Where q is the linear cost of the productive-expenditure y_n (which we will refer to as the *cost of intensification*), and $\mathcal{W}_n$ is a random production cost shock normally distributed with mean k and variance σ^2. Finally, we capture the lack of access to financial markets of the farmer by considering interests $(e^{-\alpha\tau x_n} - 1)$ to be paid at the end of each period. A farmer can only incur debt $(x_n < 0)$ by borrowing for one production cycle from within the community, and likewise can lend excess money to the community members. The exponential function is capturing the increasing marginal rates of borrowing (and decreasing marginal rates for lending). This leads to the following dynamics for the farmer's cash position:

$$x_{n+1} = x_n + \underbrace{((y_n)^\lambda \ell_n \tau) p_n}_{\substack{\text{production}\\\text{revenue}}} - \underbrace{c_n \tau}_{\text{consumption}} - \underbrace{((y_n)^\lambda \mathcal{W}_n + q y_n)\ell_n \tau}_{\substack{\text{production}\\\text{and intensification}\\\text{costs}}} - \underbrace{(e^{-\alpha\tau x_n} - 1)}_{\text{interest payments}} - \underbrace{(\ell_n^d)^+ d}_{\substack{\text{deforestation}\\\text{cost}}} \tag{2}$$

Additionally, the land expansion decision ℓ_n^d leads to the following land dynamics:

$$\ell_{n+1} = \ell_n + \ell_n^d. \tag{3}$$

Subject to these dynamics the farmer will maximize her expected discounted consumption:

$$\mathbb{E}\left[\int_0^D e^{-\tau\beta} c_{\lceil t/\tau \rceil} dt\right] = \hat{\beta}\mathbb{E}\sum_{n=1}^{N+1} e^{-n\beta\tau} c_n. \tag{4}$$

Where $\hat{\beta} = \frac{1-e^{-\beta\tau}}{\beta}$, and the terminal condition is $c_{N+1} = (x_{N+1} - (e^{-\alpha\tau x_{N+1}} - 1))/\tau$, which corresponds to the consumption of all the cash remaining net of interest payments. In the objective, β represents the farmer's discount rate. We refer to the farmer's expected discounted consumption as the farmer's *welfare*.

2.1. Modelling Assumptions

Timing of farmer's decisions. We assume that farmers decide on their consumption and productive-expenditure rates, as well as the total amount of land they will clear at the start of each period. This assumption not only helps with tractability, but is rooted in practice. Farmers often make production and consumption decisions at the time of cash inflow (see e.g., Collins et al. 2009, Duflo and Banerjee 2011).

Cleared-land production lag. The land cleared at the start of each period will only be considered productive for the next period. This assumption captures the lag between starting to clear land and harvesting from that land. This lag has two main sources: first, clearing land is usually done with manual labor which takes a considerable amount of time (Ketterings et al. 1999), second, once the crops are planted, the time until productive maturity may vary between 100 to 200 days for crops such as maize and ginger (India-Agro-Net 2021a,b) to 42 months for perennial crops such as oil palms (Verheye 2010). Although we are assuming, for simplicity of exposition, the lag to be of one period, all our results can be readily extended to a fixed arbitrary lag T (i.e., $\ell_{n+1} = \ell_n + \ell^d_{(n+1)-T}$).

Production function. We consider the effect on the yield from a production-expenditure of y to be $(y)^\lambda$, for a fixed $0 \le \lambda \le 1$. This is in line with the notion of decreasing marginal returns that are found in almost all production technologies. In particular, if we consider y to represent the total rate of labor dedicated to production, by increasing the amount of labor, the farmer can increase the total production, but the rate of increase per unit of labor will be decreasing (Shephard and Färe 1974). If we consider y to represent the rate of fertilizer application, then the function y^λ captures the yield response curve, which has been thoroughly documented to be concave, and is commonly estimated as a power function, with the most common exponents used being $\lambda = 0.5$ and $\lambda = 0.75$ (Tilman et al. 2011, Bélanger et al. 2000, Cerrato and Blackmer 1990, Hagin 1960).

Production cost shocks. We consider random production cost shocks in (1) given by $\mathcal{W}_n(y_n)^\lambda \ell_n$. A primary example of these random cost shocks is the delivery cost faced by many smallholders in frontier regions, where roads are seldom paved, leading to highly increased costs when there is enough rain to turn the dirt into mud. Additional random costs associated to bringing the product to market may be linked to higher than expected labor costs at harvesting season.

Market price process. We assume that the market price received by the farmer, p_n, is exogenously determined. This is consistent with many situation in which farmers produce commodity products, such as maize, oil-palm, or cocoa, where the price is mostly fixed by the international markets and not affected by the farmer's own production quantity. These settings are of particular importance, as many of the major documented cases of tropical deforestation are linked to the production of such crops (see Gatto et al. 2017 for a reference on oil-palm production in Indonesia, Bruun et al. 2017 for the case of maize production in the highlands of Thailand, and Kroeger et al. 2017 for an account of deforestation in the Cocoa supply chain). We will assume that $p_n \ge k$, for every n, this assumption avoids the case where production is trivially not sustainable. Additionally, we will assume that $\mathbb{E}(p_{n+1} | \sigma(\{p_i\}_{i \le n}))$ is increasing in p_n. This implies that observing higher current prices does not lead to lower expected prices in the future. This is consistent with a wide variety of stochastic processes, including any Markovian price process, as well as any submartingale adapted to $\sigma(\{p_i\}_{i \le n})$.

Negative consumption. While we allow the farmer's consumption c_n to become negative (which can be interpreted as farmers borrowing food from friends and family), we penalize this in the farmer's welfare function (4). This assumption is needed for tractability.

Exponential interest payments. Increasing interest rates for larger loans have been extensively documented in the development literature (Duflo and Banerjee 2011, Collins et al. 2009, Ghosh et al. 2000). We capture these increasing loan rates by using an exponential function $e^{-\alpha\tau x_n}$. We assume as well that no farmer would forgo current consumption in order to lend money and use the interest earned in the future (i.e., $\alpha \le (e^\beta - 1)/\tau$).

3. Results

Theorem 1 characterizes the farmer's optimal policy.

Theorem 1. *In each period n, the farmer will choose the following productive-expenditure and consumption rates, as well as total land cleared:*

$$\ell_n^d = (\hat{\ell}_{n+1} - \ell_n)^+, \tag{5}$$

$$y_n = y_n^*(\ell_n, p_n) \tag{6}$$

$$c_n = \frac{1}{\tau}\left(x_n + p_n(y_n^*)^\lambda \ell_n \tau - (q y_n^* + k(y_n^*)^\lambda)\ell_n \tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^{d*})^+ d - g_n^*\right) \tag{7}$$

Where $g_n^* = \frac{1}{\alpha\tau}\left((\ell_n(y_n^*)^\lambda \alpha\sigma\tau^2)^2/2 - \log(\frac{e^{\beta\tau}-1}{\alpha\tau})\right)$, *and* $y^*(\ell_n, p_n)$ *solves:*

$$(y^*)^{\lambda-1}\lambda\ell_n\tau(p_n-k) - (y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}) = q\ell_n\tau. \tag{8}$$

A recursive expression for $\hat{\ell}_{n+1}$ can be found in Proposition 1.

The farmer's best response is divided into three decisions per period, two production decisions (y_n, ℓ_n^d), and one consumption decision (c_n). Interestingly, the production decisions do not depend on the cash position in period n, x_n. This result can be shown to hold for any concave increasing production function and any convex decreasing interest payment function.

The land expansion decision ℓ_n^d follows a base-stock policy form, by which the farmers expand up to a land target $\hat{\ell}_{n+1}$ (and do not expand at all if this target value is below the current amount of land ℓ_n). This target $\hat{\ell}_{n+1}$ is defined in Proposition 1 in the Appendix as the land amount that equates the expected marginal future value of land to the marginal land expansion cost d. It can be shown that if we assume the lag until new land becomes productive to be T (i.e., $\ell_{n+1} = \ell_n + \ell^d_{(n+1)-T}$), then the optimal land expansion in period n would be $\ell_n^{d*} = (\hat{\ell}_{n+T} - \ell_n - \sum_{i=1}^{T-1}\ell^d_{n-i})$, where the target land $\hat{\ell}_{n+T}$ would equate the marginal deforestation cost d to the expectation in period n of the marginal value of land in all periods following the $(n+T)$-th period.

The optimal production-expenditure rate decision leads to a total production function that increases with the price p_n. Additionally, we show in 3 that the optimal production-expenditure decreases with the marginal cost of intensification q and with the variance of the production cost shock, σ^2. This latter relationship can be explained through the high interest payments the farmers face: higher levels of risk will induce lower levels of production intensification. This is consistent with the literature on technological adoption and intensification (see, e.g., Joffre et al. (2018)).

The consumption decisions imply the farmer saves in expectation exactly up to g_n^*, which is increasing in the variance of the production cost shocks. This is consistent with the empirical findings on agricultural risks and how they affect a farmer's ability to obtain food security and higher welfare levels (see e.g., Wolgin 1975). This is even more relevant than ever when facing higher climate-change related risk (Harvey et al. 2014).

Theorem 2 shows how the farmer's value function changes with the total amount of land, the intensification cost, the expected cost of production, and the variation in this cost of production.

THEOREM 2. *The farmer's value function* $J_n(x_n, \ell_n, p_n)$ *is* ***increasing*** *in* ℓ_n, p_n, *and* x_n, *and* ***decreasing*** *in* q, k, *and* σ^2, *for every* $n \in \{1, \dots, N+1\}$.

As expected, the farmer's welfare is increasing in the total amount of land, the market price, and the cash position, and decreasing in the cost of both the cost of intensification and the expected cost of production. Moreover, the higher the variation of production costs, the lower the total welfare.

In Theorem 3 we show how the optimal production-expenditure levels change as a function of the amount of land, the cost of intensification, and the variance of the production cost shocks.

THEOREM 3. *The farmer's optimal production-expenditure level* $y_n^*(\ell_n, p_n)$ *is* ***decreasing*** *in* σ^2, ℓ_n, q, α, *and* k.

We show that the production-expenditure exerted is decreasing in the production risk. High variability in production costs generate reduced consumption as well as a reduction in the total optimal production. The more subtle insight we show in Theorem 3 is that the production-expenditure level is decreasing in the amount of land: under liquidity constraints, the higher the amount of land, the less farmers can invest in increasing the productivity per unit of land. Increased intensification cost q would, as well, decrease the total production-expenditure rate, which is in line with all the literature on incentivizing technology adoptions and better farming processes, and forms the basis of most of the input-driven incentives and technological training incentives applied widely in practice (see § 1.3 for an account of several such incentive programs).

Theorem 4 shows how the land-expansion decisions are affected by the expected production cost shock, its associated variance, and the interest rate.

THEOREM 4. *The farmer's deforestation decision ℓ_n^{d*} is **decreasing** in k, σ^2, and α.*

This result shows one of the key problems of most incentives schemes with the dual aim of increasing farmer welfare and decreasing land-expansion: most factors that improve the former increase the latter. This phenomenon has been described in the Economics literature as the Jevon's paradox, and indeed occurs frequently in practice (see Alcott 2005). In our model, we can see that decreasing the expected cost of production k or the interest rate α imply an increase in deforestation pressure. Additionally, we can see that at higher levels of variability, the deforestation pressure is reduced. Not only does high variability of costs induce lower intensification, but it reduces as well the amount of land cleared. The rationale for why this happens is similar to before: at higher variability of production costs and faced with high interest rates for debt, farmers are less prone to increase their total productive land.

In Theorem 5 we show that for low enough cost q, the land-expansion pressure is actually **increasing** in q.

THEOREM 5. *There exist positive thresholds $\tilde{q}_n^L(\ell_n) \le \tilde{q}_n^H(\ell_n)$ such that farmer f's equilibrium deforestation decision ℓ_n^{d*} is **increasing** in q for $q \le \tilde{q}_n^L(\ell_n)$, and **decreasing** in q for $q \ge \tilde{q}_n^H(\ell_n)$. Moreover, $\tilde{q}_n^L(\ell_n)$ and $\tilde{q}_n^H(\ell_n)$ are increasing in σ^2, α, and ℓ_n.*

This surprising result provides a clear insight into the contradicting empirical observations on how intensification can affect deforestation (see §1.2 and § 1.3). While many subsidy programs that reduced the cost of intensification did reduce the total deforestation, many other have had the exact opposite effect. We demonstrate here that indeed the effect can go in both directions, depending on context specific parameters. This threshold behavior is driven by the combination of the liquidity constraint and the variable production costs. In particular, when reducing the cost of intensification, there are two opposing forces acting on the deforestation pressure. One the one hand, reducing q reduces the intensification cost and increases the equilibrium production-expenditure y^* (see Theorem 3), both of which makes each unit of land more valuable. On the other hand, the increased intensification implies that each unit of land will produce a higher volume, which leads to a higher variability in the production costs. Under the liquidity constraints, this increase in total variability will induce a downward pressure on land expansion. The balance between these two forces is characterized by the threshold behavior described in Theorem 5: when the variability in production costs is high, $\tilde{q}_n^L(\ell_n)$ will be high, and deforestation pressure will decrease when decreasing q for any q smaller than $\tilde{q}_n^L(\ell_n)$, but when the variability in production costs is low compared to q, q will be above $\tilde{q}_n^H(\ell_n)$, and decreasing q will incentivize deforestation.

Interestingly, Theorem 5 shows that the reducing the intensification cost q would not only cause farmers that are "better off" to reduce deforestation. Because $\tilde{q}_n^L$ and $\tilde{q}_n^H$ are increasing in both ℓ_n, and α, then both farmers with an already large productive land, and farmers that are more liquidity constrained would reduce deforestation when their cost of intensification is reduced.

The threshold result in Theorem 5, together with the insight on the role that production cost variability and liquidity constraints play, provide an explanation to the highly debated question of whether intensification causes or prevents deforestation (see §1.2). To the best of our knowledge this is the first result that presents the level of production cost risk paired with the liquidity constraints as causes for the different answers to this question observed in practice.

3.1. Discussion on different incentive schemes

From the results above, we can surmise the following insight.

Insight 1 *The only non-conditional welfare improving interventions that can decrease deforestation pressure are those that decrease the intensification cost when this cost is low enough.*

This insight is a direct corollary of Theorems 2, 4, and 5, as any reduction of either the mean of the random production cost or the variance would indeed improve the welfare of the farmer, but would as well lead to an increase in the total land cleared. In contrast, when q, the intensification cost is lower than the threshold $\tilde{q}_n^L$, lowering this cost induces both an increase of welfare and an increase of the protected forest. This insight is validated by the empirical evidence that links the reduction in transportation costs with the increase of deforestation (see Geist and Lambin (2002) for a global analysis of this, and Bruun et al. (2017) for an example of how the improved road conditions facilitated the deforestation of the highlands in northern Thailand). This reenforces the need for careful implementation of policies, because in most cases, well intentioned welfare-improving policies can have devastating effects on the preservation of natural forests if not made conditional on conservation goals.

4. Concluding Remarks

We have introduced a dynamic model of farmer operation that allowed us to study the effect of intensification promotion on deforestation. In particular, we found that there exists a non monotonic relationship between the intensification cost and the rate of deforestation: for low enough intensification costs, decreasing the cost can reduce deforestation, while for large enough costs the opposite is true. This adds a nuanced explanation to the already existing theories that try to explain the, empirically observed, varied relationship between intensification and deforestation. In particular, our main result is driven by our consideration of random production cost shocks and liquidity constrained farmers.

Our results reinforce the importance of careful study of each context before any policy implementation. Implementing a policy of intensification promotion in a setting where the average size of plots is small, the variability in costs is low, and the intensification costs are already high, may have negative effects on the forest protection. At the same time, implementing the same policy in a region where there is high variability of costs, land sizes are not too small, and the cost of intensification is not too high may actually reduce deforestation.

Finally, we believe these findings motivate future work that could empirically validate our results. Namely, showing that in settings where the reduction of the cost of intensification led to higher deforestation, the variability in costs was low enough to put the intensification cost above the theoretical threshold. And conversely, that in settings where intensification reduced land-clearing, that the intensification cost was indeed below the threshold. Estimating these thresholds in practice would require a careful collection of farm-level data, in order to understand the sources of uncertainty in the costs, as well as estimating the model's parameters.

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5. Proofs

We begin by proving a proof of Theorem 1.

Proof of Theorem 1. Recall that the problem is divided into $N+1$ periods of length τ, where the n-th period corresponds to the time $[(n-1)\tau, n\tau)$. In the terminal period, $N+1$, the farmer consumes the remaining cash position net of interest payments. In the rest of the periods, the farmer observes the price p_n, total productive land ℓ_n, and current cash position x_n, and decides on the consumption rate c_n, the production-expenditure rate y_n, and the total deforestation amount ℓ_n^d. Let $J_n(x_n, \ell_n, p_n)$ denote the farmer's value function at time $n \in \{1, \dots, N+1\}$. We show the following Proposition that proves the desired result:

Proposition 1 *For $n \in \{1, \dots, N+1\}$,*

$$J_n(x_n, \ell_n, p_n) = \frac{\hat{\beta}}{\tau}(x_n + (1 - e^{-\alpha x_n}) + f_n(p_n, \ell_n)), \tag{9}$$

$$\ell_n^{df} = (\hat{\ell}_{n+1} - \ell_n)^+, \text{ where } \hat{\ell}_{n+1} \text{ solves } \frac{e^{-\beta\tau} \partial \mathbb{E}_n f_{n+1}(p_{n+1}, l)}{\partial l} = d, \tag{10}$$

$$y_n(p_n) = y^*(\ell_n, p_n), \tag{11}$$

$$c_n^f = \frac{1}{\tau}\Big(x_n + p_n (y_n^*)^\lambda \ell_n \tau - (q y_n^* + k (y_n^*)^\lambda) \ell_n \tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^d)^+ d - g_n\Big), \tag{12}$$

where $g_n = \frac{1}{\alpha\tau}\Big(((\ell_n)^2 (y_n)^{2\lambda} \alpha^2 \sigma^2 \tau^4)/2 - \log(\frac{e^{\beta\tau} - 1}{\alpha\tau})\Big)$, $\hat{\beta} = \frac{1 - e^{-\beta\tau}}{\beta}$, $f_n(\ell_n, p_n)$ is concave and increasing in ℓ_n and increasing in p_n, the expectation $\mathbb{E}_n$ is taken conditional on the σ-algebra $\sigma(\{p_i\}_{i \le n}, \{\mathcal{W}_i\}_{i<n})$, and $y^*(\ell_n, p_n)$ solves:

$$(y^*)^{\lambda-1} \lambda \ell_n \tau (p_n - k) - (y^*)^{2\lambda - 1} \lambda \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) = q \ell_n \tau. \tag{13}$$

Proof of Proposition 1. We first show that for $n = N+1$, equation (9) holds. In this period, the farmer no longer produces, and consumes at a constant rate $c_{N+1} = (x_{N+1} - (e^{-\alpha\tau x_{N+1}} - 1))/\tau$, that leads to a value function that can be written as:

$$\begin{aligned} J_{N+1}(x_{N+1}, \ell_{N+1}, p_{N+1}) &= \int_0^\tau \left(\frac{x_{N+1} - (e^{\alpha\tau x_{N+1}} - 1)}{\tau}\right) e^{-\beta s} ds \\ &= \frac{1 - e^{-\beta\tau}}{\beta\tau}\left(x_{N+1} - (e^{-\beta\tau x_{N+1}} - 1)\right), \end{aligned}$$

that is consistent with equation (9), taking $f_{N+1}(p_{N+1}, \ell_{N+1}) = 0$, which is constant and thus concave increasing in ℓ_{N+1} and increasing in p_{N+1}.

We now proceed by induction in n, we assume the induction hypothesis for $n+1$, and consider the farmer's decision problem at time n. The farmer's value to go function at time n is given by:

$$\begin{aligned} J_n(x_n, \ell_n, p_n) &= \max_{y_n \ge 0, c_n, \ell_n^d \ge 0} \left\{ c_n \int_0^\tau e^{-\beta s} ds + e^{\beta\tau} \mathbb{E}_n \left[J_{n+1}(x_{n+1}, p_{n+1}(\mathcal{P}_{n+1}), \ell_{n+1}) \right] \right\} \\ &= \max_{y_n \ge 0, c_n, \ell_n^d \ge 0} \frac{\hat{\beta}}{\tau} \left\{ c_n \tau + e^{-\beta\tau} \mathbb{E}_n \left[x_{n+1} - (e^{-\alpha x_{n+1}} - 1) + f_{n+1}(\ell_{n+1}, p_{n+1}) \right] \right\} \end{aligned}$$

Where the second inequality is due to the inductive hypothesis. Now, we will define the following auxiliary variable:

$$g_n = x_n + p_n y_n^\lambda \ell_n \tau - c_n \tau - (q y_n + k y_n^\lambda) \ell_n \tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^d)^+ d,$$

which, using the cash dynamics for the farmer imply that

$$c_n = \frac{1}{\tau}(x_n + p_n y_n^{\lambda}\ell_n\tau - g_n - (qy_n + kr_n)\ell_n\tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^d)^+ d),$$

$$x_{n+1} = g_n - y_n^{\lambda}\ell_n\tau\sigma\varepsilon_n,$$

with $\varepsilon \sim N(0,1)$. Using these identities, we can rewrite the value to go function at time n as

$$\begin{aligned}
&= \max_{y_n\geq 0, g_n, \ell_n^d\geq 0} \frac{\hat{\beta}}{\tau}\Big\{ \quad (x_n + p_n y_n^{\lambda}\ell_n\tau - g_n - (qy_n + ky_n^{\lambda})\ell_n\tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^d)^+ d) \\
&\qquad\qquad + e^{-\beta\tau}\mathbb{E}_n\left[x_{n+1} - (e^{-\alpha\tau x_{n+1}} - 1) + f_{n+1}(\ell_n + \ell_n^d, p_{n+1})\right]\Big\} \\
&= \max_{y_n\geq 0, g_n, \ell_n^d\geq 0} \frac{\hat{\beta}}{\tau}\Big\{ \quad (x_n + p_n y_n^{\lambda}\ell_n\tau - g_n - (qy_n + ky_n^{\lambda})\ell_n\tau - (e^{-\alpha\tau x_n} - 1) - (\ell_n^d)^+ d) \\
&\qquad\qquad + e^{-\beta\tau}\mathbb{E}_n\left[g_n - y_n^{\lambda}\ell_n\tau\sigma\varepsilon_n - (e^{-\alpha\tau(g_n - y_n^{\lambda}\ell_n\tau\sigma\varepsilon_n)} - 1) + f_{n+1}(\ell_n + \ell_n^d, p_{n+1})\right]\Big\} \\
&= \frac{\hat{\beta}}{\tau}\big\{(x_n - (e^{-\alpha\tau x_n} - 1) + \max_{y_n\geq 0, g_n, \ell_n^d\geq 0} h(y_n, \ell_n^d, g_n)\big\}
\end{aligned}$$

Where

$$\begin{aligned}
h(y_n, \ell_n^d, g_n) &= y_n^{\lambda}\ell_n\tau p_n - g_n - (qy_n + ky_n^{\lambda})\ell_n\tau - (\ell_n^d)^+ d \\
&\qquad + e^{-\beta\tau}\mathbb{E}_n\left[g_n - (e^{-\alpha\tau(g_n - y_n^{\lambda}\ell_n\tau\sigma\varepsilon_n)} - 1) + f_{n+1}(\ell_n + \ell_n^d, p_{n+1})\right]. \\
&= y_n^{\lambda}\ell_n\tau p_n - g_n - (qy_n + ky_n^{\lambda})\ell_n\tau - (\ell_n^d)^+ d \\
&\qquad + e^{-\beta\tau}(g_n - (e^{-\alpha\tau g_n + (\alpha y_n^{\lambda}\ell_n\tau^2\sigma)^2/2} - 1) + \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1})).
\end{aligned}$$

This last equality follows from taking the Gaussian Moment Generating Function (recall that $\varepsilon \sim N(0,1)$):

$$\mathbb{E}_n\left[(e^{-\alpha\tau(g_n - r_n\ell_n\tau\sigma\varepsilon_n)}\right] = e^{-\alpha\tau g_n + (\alpha y_n^{\lambda}\ell_n\tau^2\sigma)^2/2}$$

We can separate $h(y_n, \ell_n^d, g_n)$ into two functions,

$$h(y_n, \ell_n^d, g_n, \ell_n, p_n) = h^1(y_n, g_n, \ell_n, p_n) + h^2(\ell_n^d, \ell_n, p_n),$$

where

$$h^1(y_n, g_n, \ell_n, p_n) = \ell_n\tau(y_n^{\lambda}(p_n - k) - y_n q) - (1 - e^{-\beta\tau})g_n - e^{-\beta\tau}(e^{-\alpha\tau g_n + (\alpha y_n^{\lambda}\ell_n\tau^2\sigma)^2/2} - 1),$$

$$h^2(\ell_n^d, \ell_n, p_n) = \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d.$$

By inductive hypothesis, we know that $f_{n+1}(\ell, p_{n+1})$ is concave and increasing in it's first argument, which implies that $h^2(\ell_n^d)$ is concave in ℓ_n^d. Hence, we can take the first order conditions to maximize $h^2(\ell_n^d)$:

$$\frac{\partial h^2(\ell_n^{d*}, \ell_n, p_n)}{\partial \ell_n^d} = 0 \Leftrightarrow \frac{e^{-\beta\tau}\partial\mathbb{E}_n f_{n+1}(\ell_n + \ell_n^{d*}, p_{n+1})}{\partial \ell_n^d} = \mathbb{1}_{\{\ell_n^{d*}\geq 0\}} d. \tag{14}$$

From equation (14) we conclude that the optimal ℓ_n^d must satisfy $\ell_n^d = (\hat{\ell}_{n+1} - \ell_n)^+$, where $\hat{\ell}_{n+1}$ solves $\frac{e^{-\beta\tau}\partial\mathbb{E}_n f_{n+1}(\ell, p_{n+1})}{\partial\ell} = d$. This proves equation (10).

In order to optimize $h^2(y_n, g_n, \ell_n, g_n)$, we will begin by finding $g_n^*(y_n)$. Notice that $h^2(y_n, g_n)$ is concave in g_n, because it is an affine function of g_n minus a convex function of g_n. Hence, we can take the first order conditions on $h^1(y_n, g_n, \ell_n, g_n)$, with respect to g_n, to obtain:

$$\frac{\partial h^1(y_n, g_n^*, \ell_n, p_n)}{\partial g_n} = 0 \Leftrightarrow exp(-\alpha\tau g_n^* + (\alpha y_n^\lambda \ell_n \tau^2 \sigma)^2/2) = \frac{e^{\beta\tau}}{\tau\alpha}. \tag{15}$$

By using equation (15), we obtain $g_n^* = \frac{1}{\alpha\tau}\left((\ell_n^2 y_n^{2\lambda}\alpha^2\sigma^2\tau^4)/2 - \log(\frac{e^{\beta\tau}-1}{\alpha\tau})\right)$, proving (12). Now, we can write $h^3(y_n, \ell_n, p_n) = \max_{g_n} h^1(y_n, g_n^*(y_n), \ell_n, p_n)$, as:

$$\begin{aligned} h^3(y_n, \ell_n, p_n) &= \ell_n\tau(y_n^\lambda(p_n - k) - y_n q) - (1 - e^{-\beta\tau})(\frac{1}{\alpha\tau}\left((\ell_n^2 y_n^{2\lambda}\alpha^2\sigma^2\tau^4)/2 - \log(\frac{e^{\beta\tau}-1}{\alpha\tau})\right)) \\ &\qquad - e^{-\beta\tau}(e^{-\alpha\tau g_n^* + (\alpha y_n^\lambda \ell_n \tau^2\sigma)^2/2} - 1) \\ &= \ell_n\tau(y_n^\lambda(p_n - k) - y_n q) - (1 - e^{-\beta\tau})\left((\ell_n^2 y_n^{2\lambda}\alpha\sigma^2\tau^3)/2 - \frac{1}{\alpha\tau}\log(\frac{e^{\beta\tau}-1}{\alpha\tau})\right) \\ &\qquad - e^{-\beta\tau}\left(\frac{e^{\beta\tau}-1}{\tau\alpha} - 1\right). \end{aligned} \tag{16}$$

Where the second equality uses the characterization in (15).

In order to maximize $h^3(y_n, \ell_n, p_n)$, we begin by taking the first order conditions to find the stationary point y_n^*:

$$\frac{\partial h^3(y_n^*, \ell_n, p_n)}{\partial y_n} = 0 \Leftrightarrow (y^*)^{\lambda-1}\lambda\ell_n\tau(p_n - k) - (y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1 - e^{-\beta\tau}) = q\ell_n\tau. \tag{17}$$

In order to show that y^* is indeed a maximum of $h^3(y_n, \ell_n, p_n)$, we show in Proposition 2 that $\frac{\partial^2 h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial y_n^2} \le 0$, for all ℓ_n and p_n, which proves (13), and gives an implicit characterization of the optimal production-expenditure $y_n^*(\ell_n, p_n)$.

Putting together the results shown above, we can write

$$\begin{aligned} J_n(x_n, \ell_n, p_n) &= \frac{\hat{\beta}}{\tau}\{(x_n - (e^{-\alpha\tau x_n} - 1) + h^3(y_n^*(\ell_n, p_n), \ell_n, p_n) + h^2(\ell_n^{d*}(\ell_n, p_n), \ell_n, p_n)\} && (18) \\ &= \frac{\hat{\beta}}{\tau}\{(x_n - (e^{-\alpha\tau x_n} - 1) + f_n(\ell_n, p_n)\}, && (19) \end{aligned}$$

where $f_n(\ell_n, p_n) = h^3(y_n^*(\ell_n, p_n), \ell_n, p_n) + h^2(\ell_n^{d*}(\ell_n, p_n), \ell_n, p_n)$. Therefore, to conclude the proof of the proposition, we need only to show that $f_n(\ell_n, p_n)$ is concave and increasing in ℓ_n, and increasing in p_n. To show this, we begin by observing that

$$h^2(\ell_n^{d*}(\ell_n, p_n), \ell_n, p_n) = \max_{\ell_n^d}[\mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d],$$

Where, by inductive hypothesis, $h^2(\ell_n^d, \ell_n, p_n) = \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d$ is jointly concave in both ℓ_n and ℓ_n^d. Thus, because partial maximization of a jointly concave function preserves concavity, $h^2(\ell_n^{d*}(\ell_n, p_n), \ell_n, p_n)$ must be concave in ℓ_n. To show that it is increasing in ℓ_n, we observe that $\max\{\ell_n, \hat{\ell}_{n+1}\}$ is increasing in ℓ_n, and $(\hat{\ell}_{n+1} - \ell_n)^+$ is decreasing in ℓ_n, this together with the inductive hypothesis gives us the results for $h^2(\ell_n^{d*}(\ell_n, p_n), \ell_n, p_n)$. That it is increasing in p_n is a consequence of the inductive hypothesis and the fact that $\mathbb{E}(p_{n+1}|\sigma(\{p_i\}_{i\le n}))$ is increasing in p_n.

Finally, we need only to prove that $h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)$ is concave increasing in ℓ_n and increasing in p_n. We proceed by considering the first and second derivative of $h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)$ with respect to ℓ_n, and the first derivative with respect to p_n.

$$\begin{aligned}
\frac{dh^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n} &= \underbrace{\frac{\partial h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial y}}_{\text{0, by definition of } y^*}\frac{dy_n^*(\ell_n,p_n)}{d\ell_n}+\frac{\partial h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial \ell_n}\\
&= \tau((y_n^*)^{\lambda}(p_n-k)-y_n^*q)-(1-e^{-\beta\tau})(\ell_n(y_n^*)^{2\lambda}\alpha\sigma^2\tau^3)\\
&= \frac{y_n^*}{\lambda\ell_n}\left[\tau\ell_n((y_n^*)^{\lambda-1}\lambda(p_n-k)-\lambda q)-(1-e^{-\beta\tau})\lambda(\ell_n^2(y_n^*)^{2\lambda-1}\alpha\sigma^2\tau^3)\right]\\
&= \frac{y_n^*}{\lambda\ell_n}\left[\tau\ell_n(y_n^*)^{\lambda-1}\lambda(p_n-k)-(1-e^{-\beta\tau})\lambda(\ell_n^2(y_n^*)^{2\lambda-1}\alpha\sigma^2\tau^3)-\lambda q\ell_n\tau\right]\\
&= \frac{y_n^*q\ell_n\tau(1-\lambda)}{\lambda\ell_n}\\
&= \frac{y_n^*q\tau(1-\lambda)}{\lambda}\geq 0.
\end{aligned}$$

The fourth equality above uses the implicit definition of y^* (17).

$$\begin{aligned}
\frac{dh^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{dp_n} &= \underbrace{\frac{\partial h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial y}}_{\text{0, by definition of } y^*}\frac{dy_n^*(\ell_n,p_n)}{dp_n}+\frac{\partial h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial p_n}\\
&= \tau\ell_n(y_n^*)^{\lambda}\geq 0.
\end{aligned}$$

This proves that $h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)$ is indeed increasing in ℓ_n and p_n. Moreover, we see that $\frac{d^2h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n^2}\leq 0$ if and only if $\frac{dy^*(\ell_n,p_n)}{d\ell_n}\leq 0$, which we prove in Proposition 3. Therefore, $f_n(\ell_n,p_n)$ is both increasing in ℓ_n and concave in ℓ_n, which completes the proof of the inductive step. □

Proposition 2 *Let $h^3(y_n,\ell_n,p_n)$ be as defined in equation (16), and $y^*(\ell_n,p_n)$ be the optimal production-expenditure level as defined by (17), then $\frac{\partial^2h^3(y_n^*,\ell_n,p_n)}{\partial y^2}\leq 0$, for any $\ell_n\geq 0$ and p_n.*

Proof. First, let us compute the first derivative $\frac{\partial h^3(y_n^*,\ell_n,p_n)}{\partial y}$:

$$\frac{\partial h^3(y,\ell_n,p_n)}{\partial y}=y^{\lambda-1}\lambda\ell_n\tau(p_n-k)-y^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})-q\ell_n\tau.$$

From here, we can compute the desired second derivative:

$$\begin{aligned}
\frac{\partial^2h^3(y_n^*,\ell_n,p_n)}{\partial y^2} &= (y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})\\
&= \frac{1}{y_n^*}[(y^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]\\
&= \frac{1}{y_n^*}[(\lambda-1)q\ell_n\tau-(y^*)^{2\lambda-1}\lambda^2\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]\leq 0.
\end{aligned}$$

Where the first equality uses the fact that $y^*=0$ is not a solution to equation (17), as long as $q>0$ (and if $q=0$, we consider the unique positive solution defined by $\left(\frac{(p_n-k)}{\ell_n\tau^2\alpha(1-e^{-\beta\tau})}\right)^{\frac{1}{\lambda}}$). Additionally, the second equality uses the definition of y^*, that implies that $(y^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k)=(\lambda-1)[y^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})+q\ell_n\tau]$. And finally, the last inequality stems from the simple observation that $\lambda\le 1$. □

Proposition 3 *Let $y_n^*(\ell_n,p_n)$ be the optimal production-expenditure level, as defined by (17), then $y_n^*(\ell_n,p_n)$ is decreasing in ℓ_n, i.e.,* $\frac{dy^*(\ell_n,p_n)}{d\ell_n}\le 0$.

Proof. We compute the derivative of $y_n^*(\ell_n,p_n)$ with respect to ℓ_n, by using the Implicit Function Theorem and the definition of y^* in (17).

$$\begin{aligned}\frac{dy^*(\ell_n,p_n)}{d\ell_n}&=-\frac{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y\partial\ell_n}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\&=-\frac{(y^*)^{\lambda-1}\lambda\tau(p_n-k)-(y^*)^{2\lambda-1}\lambda 2\ell_n\tau^3\sigma^2\alpha(1-e^{-\beta\tau})-q\tau}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\end{aligned}$$

But, by Proposition 2, we know that $\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}\le 0$, this implies that:

$$\begin{aligned}sign(\frac{dy^*(\ell_n,p_n)}{d\ell_n})&=sign((y^*)^{\lambda-1}\lambda\tau(p_n-k)-(y^*)^{2\lambda-1}\lambda 2\ell_n\tau^3\sigma^2\alpha(1-e^{-\beta\tau})-q\tau)\\&=sign(\frac{1}{\ell_n}[(y^*)^{\lambda-1}\lambda\ell_n\tau(p_n-k)-2(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})\ -\ell_n q\tau)]\\&=sign(-(y^*)^{2\lambda-1}\lambda\ell_n\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))\qquad\text{by (17)}\end{aligned}$$

Which implies that $\frac{dy^*(\ell_n,p_n)}{d\ell_n}\le 0$, proving the result. □

These propositions together finish the proof of Theorem 1. ■

We proceed to prove Theorem 2.

Proof of Theorem 2. As a consequence of the characterization proven in Theorem 1, and $f_n(\ell_n,p_n)$ being increasing in ℓ_n and p_n for every $n\in\{1,\dots,N+1\}$, we have that $J_n(x_n,\ell_n,p_n)$ must be increasing in ℓ_n and p_n. Moreover, due to this same characterizations, $J_n(x_n,\ell_n,p_n)$ is increasing in x_n if and only if $x_n+(1-e^{-\alpha x_n})$ is increasing in x_n, which can be seen by simple inspection. We need then only to prove that the value function is decreasing in q, k, and σ^2. We proceed to show by backwards induction in n that

$$\begin{aligned}\frac{\partial J_n(x_n,\ell_n,p_n)}{\partial k}&\le 0,\\\frac{\partial J_n(x_n,\ell_n,p_n)}{\partial q}&\le 0,\\\frac{\partial J_n(x_n,\ell_n,p_n)}{\partial\sigma^2}&\le 0.\end{aligned}$$

When $n=N+1$, then $J_{N+1}(x_n,\ell_n,p_n)=\frac{1-e^{-\beta\tau}}{\beta\tau}\left(x_{N+1}-(e^{-\beta\tau x_{N+1}}-1)\right)$, which implies that $\frac{\partial J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial k}=\frac{\partial J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial\alpha}=\frac{\partial J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial\sigma^2}=0.$

Now we proceed by assuming that the result holds for $n+1$, and proving that it must hold for n. Using the characterization shown in (18) in the proof of Proposition 1, we can write the value function at time n as:

$$J_n(x_n, \ell_n, p_n) = \frac{\hat{\beta}}{\tau}\big\{(x_n - (e^{-\alpha\tau x_n} - 1) + h^3(y_n^*(\ell_n, p_n), \ell_n, p_n) + \max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)\big\},$$

where

$$\begin{aligned} h^3(y_n^*(\ell_n, p_n), \ell_n, p_n) &= \ell_n \tau (y_n^*)^\lambda (p_n - k) - y_n^* q) - (1 - e^{-\beta\tau})(\ell_n^2 (y_n^*)^{2\lambda} \alpha\sigma^2\tau^3)/2 \\ &\quad - \frac{1}{\alpha\tau}\log(\frac{e^{\beta\tau} - 1}{\alpha\tau}) - e^{-\beta\tau}(\frac{e^{\beta\tau}}{\tau\alpha} - 1), \\ h^2(\ell_n^d, \ell_n, p_n) &= \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d. \end{aligned}$$

Hence, if we wish to take the derivative of the value function with respect to the parameters k, q, and σ^2, we need only consider the derivatives of $h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)$, and $\max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)$. Let us begin by considering the latter:

$$\begin{aligned} \frac{\partial \max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)}{\partial k} &= \frac{\partial \max_{\ell + n^d \geq 0} \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d}{\partial k} \\ &= \frac{\partial \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d}{\partial k}(\ell_n^{d*}, \ell_n, p_n) \qquad \text{Using Envelope Theorem.} \\ &= \mathbb{E}_n \frac{\partial f_{n+1}(\ell_n + \ell_n^d, p_{n+1})}{\partial k}(\ell_n^{d*}, \ell_n, p_n) \leq 0. \end{aligned}$$

Where the second equality is an application of the Envelope Theorem (see Milgrom and Segal 2002), and the final inequality comes from the inductive hypothesis. The same arguments prove that $\max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)$ must be decreasing in q and σ^2. It only remains to be seen that the same is true for $h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)$.

$$\begin{aligned} \frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{dk} &= \underbrace{\frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial y}}_{\text{0, by definition of } y^*} \frac{dy_n^*(\ell_n, p_n)}{dk} + \frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial k}, \\ &= -\ell_n \tau (y_n^*)^\lambda \leq 0, \end{aligned}$$

$$\begin{aligned} \frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{dq} &= \underbrace{\frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial y}}_{\text{0, by definition of } y^*} \frac{dy_n^*(\ell_n, p_n)}{dq} + \frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial q}, \\ &= -\ell_n \tau (y_n^*) \leq 0, \end{aligned}$$

$$\frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{d\sigma^2} = \underbrace{\frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial y}}_{\text{0, by definition of } y^*} \frac{dy_n^*(\ell_n, p_n)}{d\sigma^2} + \frac{\partial h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial \sigma^2},$$

$$= -\frac{1}{2}\ell_n^2\tau^3\alpha(1-e^{-\beta\tau})(y_n^*)^{2\lambda} \le 0.$$

Where all the inequality above are readily apparent. This shows that the derivative of the value function with respect to k, σ^2, and q must be negative for n, and thus concludes the inductive proof. ■

Now we will prove Theorem 3, by using the same logic as in Proposition 3.

Proof of Theorem 3. We wish to see that the optimal production-expenditure rate $y_n^*(\ell_n, p_n)$ as defined by (17), is decreasing in ℓ_n, q, k, α, and σ^2. First, notice that Proposition 3 proves already the first result. Following the same reasoning, we will compute

$$\begin{aligned}\frac{dy^*(\ell_n,p_n)}{dq} &= -\frac{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y\partial q}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= -\frac{-\ell_n\tau}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= \frac{\ell_n\tau}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \le 0.\end{aligned}$$

Where, the numerator is always positive and the denominator was proven to be negative in Proposition 2. We similarly compute the derivative of the optimal production-expenditure rate with respect to interest rate α and expected production cost k:

$$\begin{aligned}\frac{dy^*(\ell_n,p_n)}{d\alpha} &= -\frac{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y\partial \alpha}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= -\frac{-\ell_n^2\tau^3\sigma^2\lambda(1-e^{-\beta\tau})(y_n^*)^{2\lambda-1}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= \frac{\ell_n^2\tau^3\sigma^2\lambda(1-e^{-\beta\tau})(y_n^*)^{2\lambda-1}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \le 0.\end{aligned}$$

$$\begin{aligned}\frac{dy^*(\ell_n,p_n)}{dk} &= -\frac{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y\partial k}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= -\frac{-\ell_n\tau\lambda(y_n^*)^{\lambda-1}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}}\\ &= \frac{\ell_n^2\tau^3\sigma^2\lambda(1-e^{-\beta\tau})(y_n^*)^{2\lambda-1}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \le 0.\end{aligned}$$

And, as before, we see that the numerator is always positive while the denominator is always negative. Finally, we compute the derivative with respect to σ^2:

$$\begin{aligned}\frac{dy^*(\ell_n,p_n)}{d\sigma^2} &= -\frac{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y \partial \sigma^2}}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \\ &= -\frac{-(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\alpha(1-e^{-\beta\tau})}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \\ &= \frac{(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\alpha(1-e^{-\beta\tau})}{\frac{\partial^2 h^3(y^*,\ell_n,p_n)}{\partial y^2}} \le 0.\end{aligned}$$

Where, as before, the numerator is always positive and the denominator is always negative. This proves that the optimal production-expenditure rate is decreasing in ℓ_n, q, and σ^2.

■

We will now state and prove modularity results on $J_n(x_n,\ell_n,p_n)$ that will allow us to prove Theorem 4.

Proposition 4 *The value function $J_n(x_n,\ell_n,p_n)$ is sub-modular in (ℓ_n,α), (ℓ_n,k), and (ℓ_n,σ^2), for every $n\in\{1,\dots,N+1\}$.*

Proof. We will proceed to show by backwards induction in n that

$$\frac{\partial^2 J_n(x_n,\ell_n,p_n)}{\partial \ell_n \partial k} \le 0,$$

$$\frac{\partial^2 J_n(x_n,\ell_n,p_n)}{\partial \ell_n \partial \alpha} \le 0,$$

$$\frac{\partial^2 J_n(x_n,\ell_n,p_n)}{\partial \ell_n \partial \sigma^2} \le 0.$$

When $n = N+1$, then $J_{N+1}(x_n,\ell_n,p_n) = \frac{1-e^{-\beta\tau}}{\beta\tau}\left(x_{N+1} - (e^{-\beta\tau x_{N+1}} - 1)\right)$, which implies that $\frac{\partial^2 J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial \ell_{N+1}\partial k} = \frac{\partial^2 J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial \ell_{N+1}\partial \alpha} = \frac{\partial^2 J_{N+1}(x_{N+1},\ell_{N+1},p_{N+1})}{\partial \ell_{N+1}\partial \sigma^2} = 0$. Now, as in the proof of Theorem 2, we proceed with the inductive step assuming the inductive hypothesis for $n+1$ and using the characterization of the value function shown in (18). Following the argument in the proof of Theorem 2, it suffices to show that $h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)$ and $\max_{\ell_n^d\ge 0} h^2(\ell_n^d,\ell_n,p_n)$ are both sub-modular in (ℓ_n,α), (ℓ_n,k), and (ℓ_n,σ^2). For this, we consider the crossed derivatives and show that they are negative:

$$\begin{aligned}\frac{\partial^2 \max_{\ell_n^d\ge 0} h^2(\ell_n^d,\ell_n,p_n)}{\partial \ell_n \partial k} &= \frac{\partial^2 \max_{\ell+n^d\ge 0} \mathbb{E}_n f_{n+1}(\ell_n+\ell_n^d,p_{n+1}) - (\ell_n^d)^+ d}{\partial \ell_n \partial k} \\ &= \frac{\partial^2 \mathbb{E}_n f_{n+1}(\ell_n+\ell_n^d,p_{n+1}) - (\ell_n^d)^+ d}{\partial \ell_n \partial k}(\ell_n^{d*},\ell_n,p_n) && \text{Using Envelope Theorem.} \\ &= \mathbb{E}_n \frac{\partial^2 f_{n+1}(\ell_n+\ell_n^d,p_{n+1})}{\partial \ell_n \partial k}(\ell_n^{d*},\ell_n,p_n) \le 0.\end{aligned}$$

Where the second equality is an application of the Envelope Theorem and the last inequality is due to the inductive hypothesis. This same argument can be made for (ℓ_n, α), and (ℓ_n, σ^2). Thus, we need only to show that the crossed derivatives of $h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)$ are negative. In the proof of Theorem 1 we have already shown that

$$\frac{dh^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n} = \frac{y_n^* q\tau(1-\lambda)}{\lambda} \geq 0.$$

This implies that $\frac{d^2h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n dk} \leq 0$ if and only if $\frac{dy_n^*(\ell_n,p_n)}{dk} \leq 0$, and equivalently for α, and σ^2. But we have already shown that this is the case in proving Theorem 3. Hence, we have $J_n(x_n, \ell_n, p_n)$ is sub-modular in (ℓ_n, k), (ℓ_n, α), and (ℓ_n, σ^2), proving the inductive step and the proposition. □

We now prove Theorem 4 using the modularity results from Proposition 4.

Proof of Theorem 4. In Proposition 1, we prove that $\ell_n^{d*} = (\hat{\ell}_{n+1} - \ell_n)^+$, where $\hat{\ell}_{n+1}$ solves $\frac{e^{-\beta\tau}\partial \mathbb{E}_n f_{n+1}(\ell, p_{n+1})}{\partial \ell} = d$. Additionally, in Proposition 4, we showed that $J_n(x_n, \ell_n, p_n)$ is sub-modular in (ℓ_n, k), (ℓ_n, q), and (ℓ_n, σ^2), that by the characterization proven in Proposition 1 implies that $f_n(\ell_n, p_n)$ is as well for every $n \in \{1, \ldots, N+1\}$. Therefore, a simple application on Topkis' theorem (Topkis 1998) shows that $\hat{\ell}_{n+1}$ must be decreasing in k, α, and σ^2 for $n \in \{1, \ldots, N\}$, which implies that ℓ_n^{d*} is decreasing for $n \in \{1, \ldots, N\}$. Finally, ℓ_{N+1}^d is always zero by definition, which concludes the proof that ℓ_n^{d*} is decreasing in k, α, and σ^2, for $n \in \{1, \ldots, N+1\}$.

■

In order to prove Theorem 5, we proceed to show that the modularity of $J_n(x_n, \ell_n, p_n)$, with respect to (ℓ_n, q) has the same threshold behavior.

Proposition 5 *There exists positive functions $\tilde{q}_n^H(\ell_n)$ and $\tilde{q}_n^L(\ell_n)$ such that the value function $J_n(x_n, \ell_n, p_n)$ is **super-modular** in (ℓ_n, q), for $q \leq \tilde{q}_n^L(\ell_n)$, and **sub-modular** in (ℓ_n, q), for $q > \tilde{q}_n^H(\ell_n)$ for every $n \in \{1, \ldots, N+1\}$. Moreover, $\tilde{q}_n^H(\ell_n)$ and $\tilde{q}_n^H(\ell_n)$ are increasing in ℓ_n, σ^2, and α.*

Proof. As in the proof of Proposition 4, we will proceed by backwards induction in n to show that there exists a $\tilde{q}(\ell_n)$ such that

$$\frac{\partial^2 J_n(x_n, \ell_n, p_n)}{\partial \ell_n \partial q} \geq 0, \text{ if } q \leq \tilde{q}_n^L(\ell_n),$$

and

$$\frac{\partial^2 J_n(x_n, \ell_n, p_n)}{\partial \ell_n \partial q} \leq 0, \text{ if } q \geq \tilde{q}_n^H(\ell_n).$$

Moreover, $\tilde{q}_n^H(\ell_n)$ and $\tilde{q}_n^L(\ell_n)$ are increasing in ℓ_n, σ^2, and α.

When $n = N+1$, $J_{N+1}(x_n, \ell_n, p_n) = \frac{1-e^{-\beta\tau}}{\beta\tau}\left(x_{N+1} - (e^{-\beta\tau x_{N+1}} - 1)\right)$, which implies that $\frac{\partial^2 J_{N+1}(x_{N+1}, \ell_{N+1}, p_{N+1})}{\partial \ell_n \partial q} = 0$. We proceed then to the inductive step, where we will assume the result holds for $n+1$.

Using the characterization shown in (18), we can see that it suffices to prove that $h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)$ and $\max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)$ both satisfy the desired property. In particular,

$$\begin{aligned}
\frac{\partial^2 \max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)}{\partial \ell_n \partial q} &= \frac{\partial^2 \max_{\ell + n^d \geq 0} \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d}{\partial \ell_n \partial q} \\
&= \frac{\partial^2 \mathbb{E}_n f_{n+1}(\ell_n + \ell_n^d, p_{n+1}) - (\ell_n^d)^+ d}{\partial \ell_n \partial q}(\ell_n^{d*}, \ell_n, p_n) && \text{Using Envelope Theorem.}
\end{aligned}$$

$$= \mathbb{E}_n \frac{\partial^2 f_{n+1}(\ell_n + \ell_n^d, p_{n+1})}{\partial \ell_n \partial q}(\ell_n^{d*}, \ell_n, p_n).$$

Which, by the inductive hypothesis implies that $\frac{\partial^2 \max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)}{\partial \ell_n \partial q} \geq 0$, if $q \leq \tilde{q}^L_{n+1}(\max\{\ell_n, \hat{\ell}_{n+1}\})$, and $\frac{\partial^2 \max_{\ell_n^d \geq 0} h^2(\ell_n^d, \ell_n, p_n)}{\partial \ell_n \partial q} \leq 0$, if $q \geq \tilde{q}^H_{n+1}(\max\{\ell_n, \hat{\ell}_{n+1}\})$.

Now we need only to show that the same happens for $\frac{d^2 h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{d\ell_n dq}$. We have already shown in the proof of Proposition 1 that

$$\frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{d\ell_n} = \frac{y_n^* q\tau(1-\lambda)}{\lambda} \geq 0.$$

Thus, we can compute

$$\begin{aligned}
&\frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{d\ell_n dq} \\
&= \frac{\tau(1-\lambda)}{\lambda}\left(\frac{d_n y^*(\ell_n, p_n)}{dq} q + y_n^*(\ell_n, p_n)\right) \\
&= \frac{\tau(1-\lambda)}{\lambda}\left(\frac{q\ell_n\tau}{(y_n^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})} + y_n^*\right).
\end{aligned}$$

Where the second equality uses the expression for $\frac{dy^*}{dq}$ proven in Theorem 3 (combined with the explicit form of $\frac{\partial^2 h^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{\partial y^2}$ shown in Proposition 2).The first fraction is always positive (when $\lambda \leq 1$), which means we can analyze the sign of the crossed derivative above by looking at the sign of the expression in the parenthesis.

$$\begin{aligned}
&\frac{q\ell_n\tau}{(y_n^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})} + y_n^* = \\
&= \frac{q\ell_n\tau + y_n^*((y_n^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))}{(y_n^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})}
\end{aligned}$$

As proven in Proposition 2, the denominator will always be negative. This implies that

$$\begin{aligned}
&sign(\frac{dh^3(y_n^*(\ell_n, p_n), \ell_n, p_n)}{d\ell_n dq}) \\
&= -sign(q\ell_n\tau + y_n^*((y_n^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))) \\
&= -sign(q\ell_n\tau + ((y_n^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y_n^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))) \\
&= -sign(q\ell_n\tau + ((\lambda-1)q\ell_n\tau - \lambda^2(y_n^*)^{2\lambda-1}\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))) \qquad \text{by (13)} \\
&= -sign(q\ell_n\tau\lambda - \lambda^2(y_n^*)^{2\lambda-1}\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau}))) \\
&= -sign(q - \lambda(y_n^*)^{2\lambda-1}\ell_n\tau^2\sigma^2\alpha(1-e^{-\beta\tau})) \\
&= sign(\lambda(y_n^*)^{2\lambda-1}\ell_n\tau^2\sigma^2\alpha(1-e^{-\beta\tau}) - q)
\end{aligned}$$

Consider then the function $s(q) = \lambda(y^*)^{2\lambda-1}\ell_n\tau^2\sigma^2\alpha(1-e^{-\beta\tau}) - q$. We show that this function has exactly one zero, and that it takes positive values for q lower than this zero and negative values for higher qs. On one hand, $s(0) \geq 0$, because $y^* \geq 0$. In fact, it is easy to see that at $q = 0$, (17) the only non-zero solution is $y^* = \left(\frac{(p_n - k)}{\ell_n \tau^2 \alpha (1-e^{-\beta\tau})}\right)^{\frac{1}{\lambda}}$. On the other hand, we can see that $\lim_{q\to\infty} s(q) = -\infty$. To show this, consider

$$\begin{aligned}
s(q) &= \lambda (y_n^*)^{2\lambda-1} \ell_n \tau^2 \sigma^2 \alpha (1 - e^{-\beta\tau}) - q \\
&= \frac{(y_n^*)^{\lambda-1}}{\ell_n \tau} \left(\lambda (y_n^*)^{\lambda} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) - q \ell_n \tau (y^*)^{1-\lambda} \right) \\
&= \frac{(y_n^*)^{\lambda-1}}{\ell_n \tau} \left(\lambda (y_n^*)^{\lambda} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) - \left(\lambda \ell_n \tau (p_n - k) - \lambda (y_n^*)^{\lambda} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) \right) \right) \quad \text{by (17)} \\
&= \underbrace{\frac{(y_n^*)^{\lambda-1}}{\ell_n \tau}}_{\lim\limits_{q\to\infty} (y_n^*)^{\lambda-1} = \infty} \underbrace{\left(2\lambda (y_n^*)^{\lambda} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) - \lambda \ell_n \tau (p_n - k) \right)}_{\lim\limits_{q\to\infty} (y_n^*)^{\lambda} = 0}
\end{aligned}$$

Hence, $s(q)$ can be written as the product of two expressions, one converges to infinity and the other one to $-\lambda \ell_n \tau (p_n - k) \leq 0$. We can see that $\lim\limits_{q\to\infty} (y_n^*)^{\lambda-1} = \infty$ by taking the limit as q grows to infinity of equation (17):

$$\infty = \lim_{q\to\infty} q \ell_n \tau = \lim_{q\to\infty} (y_n^*)^{\lambda-1} \left(\lambda \ell_n \tau (p_n - k) - (y^{\lambda}) \lambda \ell_n^2 \tau^3 \sigma^2 (1 - e^{-\beta\tau}) \right)$$

Therefore, $\lim\limits_{q\to\infty} s(q) = -\infty$. This implies that there must exist at least one point $\hat{q}_n^z$ that satisfies

$$s(\hat{q}_n^z) = \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n \tau^2 \sigma^2 \alpha (1 - e^{-\beta\tau}) - \hat{q}_n^z = 0. \tag{20}$$

We show this point must be unique, by showing that at every such point $s'(\hat{q}_n^z) \leq 0$.

$$\begin{aligned}
\frac{ds(\hat{q}_n^z)}{dq} &= \lambda (2\lambda - 1)(y_n^*(\hat{q}_n^z))^{2\lambda-2} \frac{dy_n^*(\hat{q}_n^z)(\ell_n, p_n)}{dq} \ell_n \tau^2 \sigma^2 \alpha (1 - e^{-\beta\tau}) - 1 \\
&= \frac{\lambda (2\lambda - 1)(y_n^*(\hat{q}_n^z))^{2\lambda-2} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau})}{(y_n^*(\hat{q}_n^z))^{\lambda-2} (\lambda - 1) \lambda \ell_n \tau (p_n - k) - (2\lambda - 1)(y_n^*(\hat{q}_n^z))^{2\lambda-2} \lambda \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau})} - 1 \\
&= \frac{(y_n^*(\hat{q}_n^z))^{\lambda-2} (1 - \lambda) \lambda \ell_n \tau (p_n - k) + 2(2\lambda - 1) \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-2} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau})}{(y_n^*(\hat{q}_n^z))^{\lambda-2} (\lambda - 1) \lambda \ell_n \tau (p_n - k) - (2\lambda - 1)(y_n^*(\hat{q}_n^z))^{2\lambda-2} \lambda \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau})}
\end{aligned}$$

We know by Proposition 2 that the denominator will always be negative, hence, we have that:

$$\begin{aligned}
&sign\Big(\frac{ds(\hat{q}_n^z)}{dq} \Big) \\
&= -sign\Big((y_n^*(\hat{q}_n^z))^{\lambda-2} (1 - \lambda) \lambda \ell_n \tau (p_n - k) + 2(2\lambda - 1) \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-2} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) \Big) \\
&= -sign\Big((y_n^*(\hat{q}_n^z))^{-1} [(y_n^*(\hat{q}_n^z))^{\lambda-1} (1 - \lambda) \lambda \ell_n \tau (p_n - k) \\
&\qquad\qquad + 2(2\lambda - 1) \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau})] \Big) \\
&= -sign\Big((y_n^*(\hat{q}_n^z))^{\lambda-1} (1 - \lambda) \lambda \ell_n \tau (p_n - k) + 2(2\lambda - 1) \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1 - e^{-\beta\tau}) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= -sign\Big((1-\lambda)[\hat{q}_n^z \ell_n \tau + \lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau})] \\
&\qquad\qquad + 2(2\lambda-1)\lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau})\Big) \\
&= -sign\Big((1-\lambda)\hat{q}_n^z \ell_n \tau + (3\lambda-1)\lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau})\Big) \\
&= -sign\Big((1-\lambda)\lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau}) + (3\lambda-1)\lambda (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau})\Big) \text{by (20)} \\
&= -sign\Big(2\lambda^2 (y_n^*(\hat{q}_n^z))^{2\lambda-1} \ell_n^2 \tau^3 \sigma^2 \alpha (1-e^{-\beta\tau})\Big)
\end{aligned}
$$

And because this last expression is always positive, we have that $s'(\hat{q}_n^z) \le 0$, that implies that there can only be one such zero.

We have shown that $\frac{dh^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n dq}$ is positive when $q \le \hat{q}_n^z$, and negative when $q \ge \hat{q}_n^z$. Therefore, taking

$$\tilde{q}_n^H(\ell_n) = \max\{\tilde{q}_{n+1}^H(\max\{\ell_n, \hat{\ell}_{n+1}\}), \hat{q}_n^z\}, \tag{21}$$

$$\tilde{q}_n^L(\ell_n) = \min\{\tilde{q}_{n+1}^L(\max\{\ell_n, \hat{\ell}_{n+1}\}), \hat{q}_n^z\}, \tag{22}$$

satisfies the conditions, because if $q \le \tilde{q}_n^L(\ell_n)$, both $\frac{dh^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{d\ell_n dq}$ and $\frac{\partial^2 \max_{\ell_n^d \ge 0} h^2(\ell_n^d,\ell_n,p_n)}{\partial \ell_n \partial q}$ are positive, and if $q \ge \tilde{q}_n^H(\ell_n)$, they are both negative.

Finally, because we assumed by the induction hypothesis that $tildeq_{n+1}^H(\ell_n)$ and $\tilde{q}_{n+1}^L(\ell_n)$ are increasing in ℓ_n, α, and σ^2, we need only to check that $\hat{q}_n^z$ is increasing in these three parameters to complete the induction. Thus, we take the derivatives of this expression using the implicit function theorem:

$$\frac{d\hat{q}_n^z}{d\ell_n} = -\frac{\frac{ds(\hat{q}_n^z)}{d\ell_n}}{\frac{ds(\hat{q}_n^z)}{dq}}.$$

We showed above that $\frac{ds(\hat{q}_n^z)}{dq} \le 0$, which means that $sign(\frac{d\hat{q}_n^z}{d\ell_n}) = sign(\frac{ds(\hat{q}_n^z)}{d\ell_n})$, but

$$
\begin{aligned}
\frac{ds(q)}{d\ell_n} &= \tau^2 \sigma^2 \alpha (1-e^{-\beta\tau}) \frac{d(y^*)^{2\lambda-1}\ell_n}{d\ell_n} \\
&= \tau^2 \sigma^2 \alpha (1-e^{-\beta\tau}) \left(\ell_n (2\lambda-1)(y^*)^{2\lambda-2} \frac{dy^*}{d\ell_n} + (y^*)^{2\lambda-1}\right).
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
&sign(\frac{d\hat{q}_n^z}{d\ell_n}) \\
&= sign(\frac{ds(q)}{d\ell_n}) \\
&= sign(\ell_n(2\lambda-1)(y^*)^{2\lambda-2}\frac{dy^*}{d\ell_n} + (y^*)^{2\lambda-1}) \\
&= sign(\ell_n(2\lambda-1)\frac{dy^*}{d\ell_n} + y^*) && ((y^*)^{2\lambda-2} \ge 0) \\
&= sign(\frac{\ell_n(2\lambda-1)(y^*)^{2\lambda-1}\lambda\ell_n\tau^3\sigma^2\alpha(1-e^{-\beta\tau})}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k) - (2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]} + y^*) && \text{by Prop. 3}
\end{aligned}
$$

$$= sign(\frac{(y^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k)}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]}).$$

The final expression is always positive, because the denominator is exactly $\frac{\partial^2 h^3(y_n^*(\ell_n,p_n),\ell_n,p_n)}{\partial y^2}$, which we proved in Proposition 2 to be negative, and the numerator is always negative when $\lambda \le 1$.

Similarly, we can compute:

$$\begin{aligned}
& sign(\frac{d\hat{q}_n^z}{d\ell_n}) \\
&= sign(\frac{ds(q)}{d\alpha}) \\
&= sign(\alpha(2\lambda-1)(y^*)^{2\lambda-2}\frac{dy^*}{d\alpha}+(y^*)^{2\lambda-1}) \\
&= sign(\alpha(2\lambda-1)\frac{dy^*}{d\alpha}+y^*) && ((y^*)^{2\lambda-2}\ge 0) \\
&= sign(\frac{\alpha(2\lambda-1)(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\sigma^2(1-e^{-\beta\tau})}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]}+y^*) && \text{by Theo. 3} \\
&= sign(\frac{(y^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k)}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]}),
\end{aligned}$$

and

$$\begin{aligned}
& sign(\frac{d\hat{q}_n^z}{d\ell_n}) \\
&= sign(\frac{ds(q)}{d\sigma^2}) \\
&= sign(\sigma^2(2\lambda-1)(y^*)^{2\lambda-2}\frac{dy^*}{d\sigma^2}+(y^*)^{2\lambda-1}) \\
&= sign(\sigma^2(2\lambda-1)\frac{dy^*}{d\sigma^2}+y^*) && ((y^*)^{2\lambda-2}\ge 0) \\
&= sign(\frac{\sigma^2(2\lambda-1)(y^*)^{2\lambda-1}\lambda\ell_n^2\tau^3\alpha(1-e^{-\beta\tau})}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]}+y^*) && \text{by Theo. 3} \\
&= sign(\frac{(y^*)^{\lambda-1}(\lambda-1)\lambda\ell_n\tau(p_n-k)}{[(y^*)^{\lambda-2}(\lambda-1)\lambda\ell_n\tau(p_n-k)-(2\lambda-1)(y^*)^{2\lambda-2}\lambda\ell_n^2\tau^3\sigma^2\alpha(1-e^{-\beta\tau})]}).
\end{aligned}$$

Where in both cases we obtain the same expression as before, which is always positive. This proves that the thresholds are always increasing in ℓ_n, α, and σ^2, and concludes the inductive proof of the proposition. □

Proof of Theorem 5 Using the same arguments as in the proof of Theorem 4, and the modularity results proven in Proposition 5, we obtain the proof of this Theorem. ■